LEVEL REGULATION OF GROUND WATER IN IRRIGATION

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Level regulation of ground water is important for preventing the irrigated ground from becoming bogged up or salinated; the evaporation and the existence of a weakly permeable horizontal waterproof stratum are taken into account. The solution is found in an explicit form. It is also shown that the solution tends asymptotically either to one of the two stationary solutions or to periodic solutions which are also obtained in this paper.

1. A mathematical investigation of irrigation processes can be found in [1-5].

In [6] the asymptotic behavior was studied of the solutions of two initial-boundary value problems for the heat-conduction equation with finite time, a nonlinear right-hand side, and with nonlinear boundary condition. It was shown that the solution of each problem,depending on the values of the initial function and of the constants appearing in the assumptions of the problem,either tends to a stable stationary solution of the heat-conduction equations with boundary conditions or to a periodic solution of the corresponding problem. Solutions of these problems can be classified in four groups.

By using the asymptotics of the solution of the first and the second boundary-value problem for parabolic equations [7] it can be shown that the four-group classification also holds for a wider class of problems.

The solution of one initial-boundary value problem was obtained in [8]. The variation with time of the level of ground water for irrigation with evaporation can be solved under the assumption that the surface of the ground water is slightly curved and that of the impermeable waterproof stratum is horizontal.

In practice one often encounters cases where a waterproof stratum is hardly permeable. The stratum is assumed to be horizontal, and of constant depth M_0.

It is assumed that the ground water occupies an area between two channels or drains $0 < x < l$ in which the water levels are $H_1(t)$ or $H_2(t)$. The following method for regulating the level of ground water by irrigation is considered: when the level measured at the point $0 < x^\circ < l$ reaches the value h_*, then the irrigation which takes place with intensity me is discontinued and it is started again when h becomes $h_{**} < h_*$.

It is assumed that the evaporation intensity is given by $m\,(d_1h + d_2)$.

One should assume that either $d_1 = 0$, $d_2 > 0$ or $d_1 > 0$, $d_2 < 0$. In the latter case the obtained results are correct for ground-water levels which are not below $-d_2/d_1$.

One is also able to take approximately into account the vertical pumping of the ground water by the holes of the vertical drainage being "smeared" over the entire region $0 < x < l$ between drains and by converting the boundary conditions at the holes into a differential equation by adding fictitious "evaporation" of intensity md_3, $d_3 = \text{const}$ (see, for example, [5]).

The problem boils down to the finding of a solution of the inhomogeneous heat-conduction equation

$$\frac{\partial h}{\partial t} = a^2 \frac{\partial^2 h}{\partial x^2} - b(h - H) + F[h(x^\circ, t)] \tag{1.1}$$

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$$F[h(x^\circ, t)] = \begin{cases} c & \text{for} \quad h(x^\circ, t) < h_* \\ -d & \text{for} \quad h(x^\circ, t) > h_{**} \end{cases} \tag{1.2}$$

with the boundary conditions

$$h(0, t) = H_1(t), \quad h(l, t) = H_2(t) \tag{1.3}$$

and the initial condition

$$h(x, 0) = \varphi(x) \tag{1.4}$$

The following notation is used in Eq. (1.2):

$$c = e - d_1 H - d_4, \quad d = d_4 + d_1 H, \quad d_4 = d_2 + d_3$$
$$a^2 = kH_0 / m, \quad b = k_0 / M_0 m + d_1$$

where k is the filtration coefficient; k_0 is the filtration coefficient of weakly permeable waterproof stratum; m is the porosity; (k_0/M_0) (H−h) is the seepage rate through waterproof layer; H_0 is the mean value of thickness of the water-bearing portion of the stratum.

It is assumed that time changes of the water levels in the channels H_1 and H_2 can be ignored; they appear in the boundary conditions (1.3); thus H_1=const, H_2=const.

Thus Eqs. (1.1), (1.2) together with the conditions (1.3) have stable and stationary solutions $v_0(x)$ and $w_0(x)$ given by

$$\begin{aligned} v_0(x) &= H + \frac{c}{b} + C_1 \operatorname{sh} \frac{\sqrt{b}}{a} x + C_2 \operatorname{sh} \frac{\sqrt{b}}{a}(x - l) \\ w_0(x) &= H - \frac{d}{b} + C_3 \operatorname{sh} \frac{\sqrt{b}}{a} x + C_4 \operatorname{sh} \frac{\sqrt{b}}{a}(x - l) \end{aligned} \tag{1.5}$$

$$\begin{aligned} C_1 &= \left(H_2 - H - \frac{c}{b}\right)\left(\operatorname{sh} \frac{\sqrt{b}}{a} l\right)^{-1} \\ C_2 &= -\left(H_1 - H - \frac{c}{b}\right)\left(\operatorname{sh} \frac{\sqrt{b}}{a} l\right)^{-1} \end{aligned} \tag{1.6}$$

The expressions for C_3 and C_4 are obtained by replacing c/b by the quantity −d/b in (1.6).

The periodic solution is given by (here T_1 is the duration of the irrigation stage, and T is the oscillation period)

$$\begin{aligned} h(x, t) &= H_1 + \frac{(H_2 - H_1)x}{l} + u_i(x, t) \qquad (i = 1, 2) \\ u_1(x, t) &= v(x) + \sum_{n=1}^{\infty} C_n \exp(-\lambda_n^2 t) \sin \frac{\pi n x}{l} \qquad (0 \leqslant t \leqslant T_1) \\ u_2(x, t) &= w(x) + \sum_{n=1}^{\infty} D_n \exp[-\lambda_n^2 (t - T_1)] \sin \frac{\pi n x}{l} \qquad (T_1 \leqslant t \leqslant T) \\ \lambda_n^2 &= \frac{\pi^2 a^2 n^2}{l^2} + b, \quad C_n = \frac{-\theta_n (1 - \gamma_n)}{1 - \delta_n}, \quad D_n = \frac{\theta_n (1 - \beta_n)}{1 - \delta_n} \\ \beta_n &= \exp(-\lambda_n^2 T_1), \quad \gamma_n = \exp[-\lambda_n^2 (T - T_1)], \\ \delta_n &= \exp(-\lambda_n^2 T) \\ \theta_n &= v_n - w_n = \frac{2(c + d)}{b} [1 - (-1)^n] \left\{\frac{1}{\pi n} - \frac{\pi n a^2}{l^2} \frac{1}{\lambda_n^2}\right\} \\ v(x) &= v_0(x) - H_1 - (H_2 - H_1)x/l, \\ w(x) &= w_0(x) - H_1 - (H_2 - H_1)x/l \end{aligned} \tag{1.7}$$

In the above, v_n and w_n are Fourier coefficients of the functions v(x) and w(x), respectively. Equations (1.5) and (1.7) yield for $0 < x^\circ < l$ the inequality $v(x^\circ) - w(x^\circ) > 0$. The quantities T_1 and $T_2 = T - T_1$ are found as the smallest roots of the system of two equations

$$\varphi(T_1,T_2)\equiv-\sum_{n=1}^{\infty}\theta_n\frac{\beta_n(1-\gamma_n)}{1-\delta_n}\sin\frac{\pi n x^\circ}{l}=u_*-v(x^\circ)$$

$$\psi(T_1,T_2)\equiv\sum_{n=0}^{\infty}\theta_n\frac{\gamma_n(1-\beta_n)}{1-\delta_n}\sin\frac{\pi n x^\circ}{l}=u_{**}-w(x^\circ) \tag{1.8}$$

$$\left(u_*=h_*-H_1-\frac{(H_2-H_1)x^\circ}{l},\quad u_{**}=h_{**}-H_1-\frac{(H_2-H_1)}{l}x^\circ\right)$$

For $T_1=T_2=0$ the functions $\varphi(T_1,T_2)$ and $\psi(T_1,T_2)$ are not continuous. It can be shown from the functions $U_*=\varphi(T_1,T_2)$ and $U_{**}=\psi(T_1,T_2)$ given by (1.8) that a pair of values $w(x^\circ)-v(x^\circ)<U_*<0$ and $0<U_{**}<v(x^\circ)-w(x^\circ)$ is uniquely associated with a pair of values T_1 and T_2, so that T_1 and T_2 do not vanish simultaneously.

Thus, with any pair of values u_* and u_{**} such that $w(x^\circ)<u_{**}<u_*<v(x^\circ)$ there is associated at least one pair of the values $T_1\neq 0$, $T_2\neq 0$. If the latter is not a unique pair, the smallest possible T_1 and T_2 is adopted.

If instead of considering the periodic solution (1.7) one considers now a periodic solution consisting of 2m portions, $u_{2j+1}(x,t)$, $u_{2j+2}(x,t)$ $(j=0,1,2,\ldots,m-1)$, then it can be seen that all portions either with odd or even subscripts are equal, that is, $(u_{2j+1}(x,t)=u_1(x,t)$, $u_{2j+1}(x,t)=u_2(x,t)$, $j=1,2,\ldots,m-1)$. This implies the uniqueness of the periodic solution (1.7).

Periodic solutions of parabolic equations with nonlinear right-hand sides were analyzed in [9-11].

2. The initial-boundary value problem (1.1)-(1.4) is considered, and it is now assumed that the inequalities

$$u_*-v(x^\circ)<0,\quad u_{**}-w(x^\circ)>0$$

hold where $\varphi(x^\circ)<u_*$ (the case of $\varphi(x^\circ)>u_*$ can be similarly analyzed). The solution of the latter problem is

$$u_1^{(i+1)}(x,t)=v(x)+\sum_{n=1}^{\infty}C_n^{(i+1)}\exp\left\{-\lambda_n^2\left(t-\sum_{j=0}^{i}T^{(j)}\right)\right\}\sin\frac{\pi n x}{l}$$

$$\left(\sum_{j=0}^{i}T^{(j)}\leqslant t\leqslant\sum_{j=0}^{i}T^{(j)}+T_1^{(i+1)},\quad i=0,1,2,3,\ldots,T^{(0)}=0\right)$$

$$u_2^{(i+1)}(x,t)=w(x)+\sum_{n=1}^{\infty}D_n^{(i+1)}\exp\left\{-\lambda_n^2\left(t-\sum_{j=0}^{i}T^{(j)}-T_1^{(i+1)}\right)\right\}\sin\frac{\pi n x}{l} \tag{2.1}$$

$$\left(\sum_{j=0}^{i}T^{(j)}+T_1^{(i+1)}\leqslant t\leqslant\sum_{j=0}^{i+1}T^{(j)}\right)$$

In the above $C_n^{(1)}$ are the Fourier coefficients of the function $\varphi(x)-v(x)$, $T_1^{(i)}$ and $T^{(i)}$ are the smallest roots of the equations

$$u_1^{(i+1)}\left(x^\circ,\sum_{j=0}^{i}T^{(j)}+T_1^{(i+1)}\right)=u_*,\quad u_2^{(i+1)}\left(x^\circ,\sum_{j=0}^{i+1}T^{(j)}\right)=u_{**} \tag{2.2}$$

The following relations can be obtained from Eqs. (2.1) and (2.2):

$$u_*-v(x^\circ)=\sum_{n=1}^{\infty}C_n^{(i+1)}\beta_n^{(i+1)}\sin\frac{\pi n x^\circ}{l}\quad(i=0,1,2,\ldots)$$

$$u_{**}-w(x^\circ)=\sum_{n=1}^{\infty}D_n^{(i+1)}\gamma_n^{(i+1)}\sin\frac{\pi n x^\circ}{l}\quad(i=0,1,2,\ldots) \tag{2.3}$$

In the above one has

$$\beta_n^{(i)}=\exp[-\lambda_n^2T_1^{(i)}],\qquad \gamma_n^{(i)}=\exp\{-\lambda_n^2[T^{(i)}-T_1^{(i)}]\}$$

The equalities

$$u_1^{(i+1)}\left(x, \sum_{j=0}^{i} T^{(j)}\right) = u_2^{(i)}\left(x, \sum_{j=0}^{i} T^{(j)}\right)$$
$$u_2^{(i+1)}\left(x, \sum_{j=0}^{i} T^{(j)} + T_1^{(i+1)}\right) = u_1^{(i+1)}\left(x, \sum_{j=0}^{i} T^{(j)} + T_1^{(i+1)}\right) \tag{2.4}$$

imply the relations

$$C_n^{(i+2)} = -\theta_n + \gamma_n^{(i+1)} D_n^{(i+1)}, \qquad D_n^{(i+1)} = \theta_n + \beta_n^{(i+1)} C_n^{(i+1)} \tag{2.5}$$

The following equalities follow from Eqs. (2.5):

$$C_n^{(i+1)} = -\theta_n \Gamma_n^{(i,k)} + \gamma_n^{(i)} \beta_n^{(i)} \gamma_n^{(i-1)} \beta_n^{(i-1)} \ldots \gamma_n^{(i-k)} \beta_n^{(i-k)} C_n^{(i-k)} \quad (i \geqslant 1)$$
$$D_n^{(i+1)} = \theta_n B_n^{(i+1,k+1)} + \beta_n^{(i+1)} \gamma_n^{(i)} \beta_n^{(i)} \ldots \gamma_n^{(i-k)} \beta_n^{(i-k)} C_n^{(i-k)} \quad (i \geqslant 0)$$
$$(\Gamma_n^{(i,k)} = 1 - \gamma_n^{(i)} + \gamma_n^{(i)} \beta_n^{(i)} - \gamma_n^{(i)} \beta_n^{(i)} \gamma_n^{(i-1)} + \ldots - \gamma_n^{(i)} \beta_n^{(i)} \ldots \beta_n^{(i-k+1)} \gamma_n^{(i-k)} \tag{2.6}$$
$$B_n^{(i,k)} = 1 - \beta_n^{(i)} + \beta_n^{(i)} \gamma_n^{(i-1)} - \beta_n^{(i)} \gamma_n^{(i-1)} \beta_n^{(i-1)} + \ldots + \beta_n^{(i)} \gamma_n^{(i-1)} \ldots \beta_n^{(i-k+1)} \gamma_n^{(i-k)})$$

In Eqs. (2.6) k may assume any integer value $k \geq 1$.

After transformations and using (2.1) and (2.6) Eqs. (2.2) assume the form

$$\beta_1^{(i+1)} = \varphi(\beta_1^{(i+1)}, \gamma_1^{(i)}, \beta_1^{(i)}, \ldots, \beta_n^{(i-k+1)}, \gamma_1^{(i-k)}, C_1'^{(i-k)}, C_2'^{(i-k)}, \ldots, C_n'^{(i-k)}, \ldots)$$
$$\gamma_1^{(i+1)} = \psi(\gamma_1^{(i+1)}, \beta_1^{(i+1)}, \gamma_1^{(i)}, \ldots, \beta_1^{(i-k+1)}, \gamma_1^{(i-k)},$$
$$C_1''^{(i-k)}, C_2''^{(i-k)}, \ldots, C_n''^{(i-k)}, \ldots) \tag{2.7}$$

for arbitrarily large value of i; in the above the following notation was introduced:

$$\varphi(\beta_1^{(i+1)}, \gamma_1^{(i)}, \beta_1^{(i)}, \ldots, \beta_1^{(i-k+1)}, \gamma_1^{(i-k)}, C_1'^{(i-k)}, C_2'^{(i-k)}, \ldots, C_n'^{(i-k)}, \ldots) =$$
$$= \beta_1^{(0)} + \sum_{n=3}^{\infty} A_n \{\beta_n^{(i+1)} \Gamma_n^{(i,k)} + \beta_1^{(i+1)} \gamma_n^{(i)} B_n^{(i,k)}\} +$$
$$+ \sum_{n=1}^{\infty} (\beta_1^{(i+1)} - \beta_n^{(i+1)}) \gamma_n^{(i)} \beta_n^{(i)} \ldots \beta_n^{(i-k+1)} \gamma_n^{(i-k)} C_n'^{(i-k)} \sin \pi n x^\circ / l \tag{2.8}$$

$$\psi(\gamma_1^{(i+1)}, \beta_1^{(i+1)}, \gamma_1^{(i)}, \ldots, \beta_1^{(i-k+1)}, \gamma_1^{(i-k)}, C_1''^{(i-k)}, C_2''^{(i-k)}, \ldots, C_n''^{(i-k)}, \ldots) =$$
$$= \gamma_1^{(0)} + \sum_{n=3}^{\infty} B_n \{\gamma_n^{(i+1)} B_n^{(i+1,k+1)} + \gamma_1^{(i+1)} \beta_n^{(i+1)} \Gamma_n^{(i,k)}\} +$$
$$+ \sum_{n=1}^{\infty} (\gamma_1^{(i+1)} - \gamma_n^{(i+1)}) \beta_n^{(i+1)} \gamma_n^{(i)} \beta_n^{(i)} \ldots \beta_n^{(i-k)} C_n''^{(i-k)} \sin \pi n x^\circ / l \tag{2.9}$$

$$\beta_1^{(0)} = q[u_* - v(x^\circ)], \quad \gamma_1^{(0)} = p[u_{**} - w(x^\circ)],$$
$$A_n = q\theta_n \sin \pi n x^\circ / l$$
$$B_n = -p\theta_n \sin \frac{\pi n x^\circ}{l}, \quad C_n'^{(i-k)} = qC_n^{(i-k)}, \quad C_n''^{(i-k)} = pC_n^{(i-k)}$$
$$p = [u_* - v(x^\circ) + \theta_1 \sin \pi x^\circ / l]^{-1}$$
$$q = [u_{**} - w(x^\circ) - \theta_1 \sin \pi x^\circ / l]^{-1} \tag{2.10}$$

In the case under consideration the following inequalities are satisfied:

$$u_* - v(x^\circ) < 0, \qquad u_{**} - w(x^\circ) > 0 \tag{2.11}$$

It follows from Eqs. (2.8)-(2.11) that Eqs. (2.7) have at least one solution for any values of $\beta_1^{(0)}$ and $\gamma_1^{(0)}$.

If there are several solutions of Eqs. (2.7), say $-\beta_1^{(i+1)}$ and $\gamma_1^{(i+1)}$, then the highest of these values is adopted.

It can be shown that the following are valid:

$$\lim_{i\to\infty}\beta_1^{(i+1)}=\beta_1,\quad \lim_{i\to\infty}\gamma_1^{(i+1)}=\gamma_1 \tag{2.12}$$

where $\beta_1=\exp(-\lambda_1^2 T_1)$, $\gamma_1=\exp[-\lambda_1^2(T-T_1)]$ for the periodic solution (2.7). The values $\beta_1^{(j)}$ and $\gamma_1^{(j)}$ which can be determined from Eqs. (2.7) and (2.8) are bounded if $u_*-u_{**}>\vartheta>0$

$$\beta_{\min}<\beta_1^{(j)}<\beta_{\max},\quad \gamma_{\min}<\gamma_1^{(j)}<\gamma_{\max} \tag{2.13}$$

The constants $\beta_{\min}$, $\gamma_{\min}$, $\beta_{\max}$, $\gamma_{\max}$ can be determined from equations which are similar to (2.7) and (2.8) by using successive approximations.

3. By introducing the notation

$$\begin{aligned}
&f(\beta_1^{(i)},\gamma_1^{(i-1)},\beta_1^{(i-1)},\ldots,\gamma_1^{(2)},\beta_1^{(2)},\gamma_1^{(1)})=\sum_{n=3}^{\infty}A_n\{(\beta_1^{(i)})^{\mu_n}\times\\
&\times[1-(\gamma_n^{(i-1)})^{\mu_n}+(\gamma_1^{(i-1)}\beta_1^{(i-1)})^{\mu_n}-(\gamma_1^{(i-1)}\beta_1^{(i-1)}\gamma_1^{(i-2)})^{\mu_n}+\ldots\\
&\ldots-(\gamma_1^{(i-1)}\beta_1^{(i-1)}\ldots\beta_1^{(2)}\gamma_1^{(1)})^{\mu_n}]+\beta_1^{(i)}(\gamma_1^{(i-1)})^{\mu_n}[1-(\beta_1^{(i-1)})^{\mu_n}+\\
&+(\beta_1^{(i-1)}\gamma_1^{(i-2)})^{\mu_n}+\ldots+(\beta_1^{(i-1)}\gamma_1^{(i-2)}\ldots\beta_1^{(2)}\gamma_1^{(1)})^{\mu_n}]\}\\
&\zeta(\beta_1^{(i+1)},\gamma_1^{(i)},\beta_1^{(i)},\ldots,\beta_1^{(2)},\gamma_1^{(1)},\beta_1^{(1)},C_1'^{(1)},C_2'^{(1)},\ldots,C_j'^{(1)},\ldots)=\sum_{n=2}^{\infty}\{[\beta_1^{(i+1)}-(\beta_1^{(i+1)})^{\mu_n}](\gamma_1^{(i)})^{\mu_n}(\beta_1^{(i)})^{\mu_n}\\
&-[\beta_1^{(i)}-(\beta_1^{(i)})^{\mu_n}]\}(\gamma_1^{(i-1)}\beta_1^{(i-1)}\ldots\gamma_1^{(1)}\beta_1^{(1)})^{\mu_n}C_n'\sin\pi nx^\circ/l+\\
&+\sum_{n=3}^{\infty}A_n[(\beta_1^{(i-1)})^{\mu_n}-\beta_1^{(i+1)}][\gamma_1^{(i)}\beta_1^{(i)}\ldots\gamma_1^{(2)}\beta_1^{(2)}]^{\mu_n}[1-(\gamma_1^{(i)})^{\mu_n}]
\end{aligned} \tag{3.1}$$

one finds from Eqs. (2.7) and (2.8) the following expression for the difference $\beta_1^{(i+1)}-\beta_1^{(i)}$:

$$\begin{aligned}
&\beta_1^{(i+1)}-\beta_1^{(i)}=f(\beta_1^{(i+1)},\gamma_1^{(i)},\beta_1^{(i)},\ldots,\gamma_1^{(3)},\beta_1^{(3)},\gamma_1^{(2)})-\\
&-f(\beta_1^{(i)},\gamma_1^{(i-1)},\beta_1^{(i-1)},\ldots,\gamma_1^{(2)},\beta_1^{(2)},\gamma_1^{(1)})+\\
&+\zeta(\beta_1^{(i+1)},\gamma_1^{(i)},\ldots,\beta_1^{(1)},C_1'^{(1)},\ldots,C_j'^{(1)},\ldots)
\end{aligned} \tag{3.2}$$

Moreover, one can write the equation

$$\begin{aligned}
&f(\beta_1^{(i+1)},\gamma_1^{(i)},\beta_1^{(i)},\ldots,\gamma_1^{(3)},\beta_1^{(3)},\gamma_1^{(2)})-\\
&-f(\beta_1^{(i)},\gamma_1^{(i-1)},\beta_1^{(i-1)},\ldots,\gamma_1^{(2)},\beta_1^{(2)},\gamma_1^{(1)})=\\
&=\xi_1^{(i)}(\beta_1^{(i+1)}-\beta_1^{(i)})+\eta_1^{(i-1)}(\gamma_1^{(i)}-\gamma_1^{(i-1)})+\xi_1^{(i-1)}(\beta_1^{(i)}-\beta_1^{(i-1)})+\\
&+\eta_1^{(i-2)}(\gamma_1^{(i-1)}-\gamma_1^{(i-2)})+\ldots+\xi_1^{(j)}(\beta_1^{(j+1)}-\beta_1^{(j)})+\\
&+\eta_1^{(j-1)}(\gamma_1^{(j)}-\gamma_1^{(j-1)})+\ldots+\eta_1^{(1)}(\gamma_1^{(2)}-\gamma_1^{(1)})
\end{aligned} \tag{3.3}$$

In the above $\xi_1^{(j)}$ ($j=2, 3, \ldots, i$) and $\eta_1^{(j)}$ ($j=1, 2, \ldots, i-1$) are the mean values of the derivatives $\partial f/\partial\beta_1^{(j)}$ and $\partial f/\partial\gamma_1^{(j)}$; they can be found by using Eq. (3.1).

An estimate can be found for the quantities $\xi_1^{(j)}$ and $\eta_1^{(j)}$ by using the inequalities (2.13).

The following notation is introduced:

$$\begin{aligned}
&q=(\beta_{\max}\gamma_{\max})^{\mu_3}\\
&\alpha=\varkappa\sum_{n=3}^{\infty}\mu_n|A_n|(\beta_{\max}-\beta_{\min}^{\mu_n})\gamma_{\max}^{\mu_n-1}\\
&\beta=\chi\sum_{n=3}^{\infty}\mu_n|A_n|(\beta_{\max}-\beta_{\min}^{\mu_n})\beta_{\max}^{\mu_n-1}\gamma_{\max}^{\mu_n}\\
&\varkappa_j^{(n)}=B_n^{(j,j-1)},\quad \chi_j^{(n)}=\Gamma_n^{(j-1,j-2)},\quad \varkappa=\max\varkappa_j^{(n)},\quad \chi=\max\chi_j^{(n)}
\end{aligned} \tag{3.4}$$

It is found from (3.1)-(3.4) that

$$\begin{aligned}
&|\xi_1^{(i-1)}|<\beta,\quad |\xi_1^{(j)}|<q^{i-1-j}\beta,\quad |\xi_1^{(j-1)}|<q|\xi_1^{(j)}|\\
&|\eta_1^{(i-1)}|<\alpha,\quad |\eta_1^{(j)}|<q^{(i-1-j)}\alpha,\quad |\eta_1^{(j-1)}|<q|\eta_1^{(j)}|
\end{aligned} \tag{3.5}$$

Let $|\xi_1^{(i)}| < E$; then one finds from (3.3) and (3.5) the inequality

$$|\beta_1^{(i+1)} - \beta_1^{(i)}| < E|\beta_1^{(i+1)} - \beta_1^{(i)}| + \alpha|\gamma_1^{(i)} - \gamma_1^{(i-1)}| + \\ + \beta|\beta_1^{(i)} - \beta_1^{(i-1)}| + \alpha q|\gamma_1^{(i-1)} - \gamma_1^{(i-2)}| + \beta q|\beta_1^{(i-1)} - \beta_1^{(i-2)}| + \ldots \\ \ldots + \beta q^{i-3}|\beta_1^{(3)} - \beta_1^{(2)}| + \alpha q^{i-2}|\gamma_1^{(2)} - \gamma_1^{(1)}| + \sigma q^{f(i-1)} \quad (f = \mu_2/\mu_3) \\ E = \sum_{n=3}^{\infty} \mu_n |A_n| \beta_{\max}^{\mu_n - 1} + \varkappa \sum_{n=3}^{\infty} |A_n| \gamma_{\max}^{\mu_n} \tag{3.6}$$

The following notation was introduced in (3.6):

$$\sigma = (\beta_{\max} - \beta_{\min}) \sum_{n=3}^{\infty} |C_n'^{(1)}| \left|\sin \frac{\pi n x^\circ}{l}\right| q^{(\mu_n - \mu_3)/\mu_3} + \beta_{\max} \sum_{n=3}^{\infty} |A_n| \tag{3.7}$$

and the following estimate was used:

$$|\zeta| < \sigma q^{f(i-1)}$$

For j = 1, 2, ..., i let the following inequalities hold;

$$|\gamma_1^{(j)} - \gamma_1^{(j-1)}| < \varepsilon_1, \quad |\beta_1^{(j)} - \beta_1^{(j-1)}| < \varepsilon_2 \tag{3.8}$$

Equation (3.6) together with (3.8) yields

$$|\beta_1^{(i+1)} - \beta_1^{(i)}| < \frac{\alpha\varepsilon_1 + \beta\varepsilon_2}{(1-E)(1-q)} + Dq^{f(i-2)} \\ \left(D = \left\{\sigma q^f + \alpha\varepsilon_1 + \frac{(\alpha\varepsilon_1 + \beta\varepsilon_2)}{1-q}\right\} \frac{1}{(1-E)}\right) \tag{3.9}$$

A similar formula is also valid for the difference $|\gamma_1^{(i+1)} - \gamma_1^{(i)}|$.

Let $\varepsilon = \max(\varepsilon_1, \varepsilon_2)$.

Let there exist a value $0 < \lambda_0 < 1$ such that the following inequalities hold:

$$\frac{\alpha\varepsilon_1 + \beta\varepsilon_2}{(1-E)(1-q)} \leqslant \lambda_0\varepsilon, \quad \frac{\gamma\varepsilon_1 + \delta\varepsilon_2}{(1-F)(1-q)} \leqslant \lambda_0\varepsilon \tag{3.10}$$

It will be shown that for $i \to \infty$ the quantities $\beta_1^{(i)}$ and $\gamma_1^{(i)}$, which are determined by Eqs. (2.7)-(2.9), approach the limits β_1 and γ_1. To this end it suffices to show that the differences $|\beta_1^{(i+p)} - \beta_1^{(i)}|$ and $|\gamma_1^{(i+p)} - \gamma_1^{(i)}|$ approach zero as $i \to \infty$. Indeed, it was previously shown that the quantities are bounded for all i, that is, that the quantities $\beta_1^{(i)}$ and $\gamma_1^{(i)}$ exist.

4. It is assumed that i is a fixed and suitably large integer. It will be shown that

$$|\beta_1^{(j+1)} - \beta_1^{(j)}| < \varepsilon\lambda^{n-1}, \quad |\gamma_1^{(j+1)} - \gamma_1^{(j)}| < \varepsilon\lambda^{n-1} \tag{4.1}$$

for

$$j = \frac{n(n+1)}{2} i - 1, \quad \frac{n(n+1)}{2} i - 2, \ldots, \quad \frac{(n-1)n}{2} i$$

For j = 1, 2, ..., i one has by assumption

$$|\beta_1^{(j+1)} - \beta_1^{(j)}| < \varepsilon, \quad |\gamma_1^{(j+1)} - \gamma_1^{(j)}| < \varepsilon \quad (n = 1)$$

It follows from the relations (3.9) that for j = i, i + 1, ..., 3i − 1

$$|\beta_1^{(j+1)} - \beta_1^{(j)}| < \lambda_0\varepsilon + Dq'^{i-2}, \quad |\gamma_1^{(j+1)} - \gamma_1^{(j)}| < \lambda_0\varepsilon + Gq'^{i-2} \quad (q' = q^f) \tag{4.2}$$

The inequalities

$$|\beta_1^{(j+1)} - \beta_1^{(j)}| < \lambda\varepsilon, \quad |\gamma_1^{(j+1)} - \gamma_1^{(j)}| < \lambda\varepsilon \quad (0 < \lambda < 1) \tag{4.3}$$

are satisfied if

$$\lambda_0\varepsilon + Mq'^{i-2} < \lambda\varepsilon \quad (M = \max(D, G)) \tag{4.4}$$

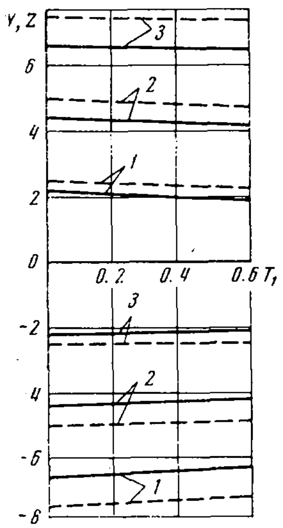

Fig. 1

For sufficiently large i one has

$$q'^{i-2} < (\lambda - \lambda_0) M^{-1} \varepsilon$$

where $0 < \lambda < 1$ and $0 < \lambda_0 < \lambda < 1$; thus the inequality (4.4) holds.

It can be shown similarly that the inequalities (4.1) hold for all j.

With the aid of (4.1) one can find an estimate for the difference $|\beta_1^{(j+p)}$ $\beta_1^{(j+1)}|$ for any $p = 1, 2, \ldots$.

By using the inequality

$$|\beta_1^{(j+p)} - \beta_1^{(j+1)}| \leqslant |\beta_1^{(j+p)} - \beta_1^{(j+p-1)}| + |\beta_1^{(j+p-1)} - \beta_1^{(j+p-2)}| + \ldots + |\beta_1^{(j+2)} - \beta_1^{(j+1)}|$$

it can be shown that the necessary and sufficient condition for the existence of the limit β_1 of the sequence $\beta_1^{(j)}$ is satisfied. The existence of the limit γ_1 for the sequence $\gamma_1^{(j)}$ is demonstrated similarly.

Let us suppose now that the water level in channels or drains are time dependent $H_i = H_i(t)$ $(i = 1, 2)$ and that

$$\lim_{t \to \infty} H_i(t) = H_{i\infty}, \quad \lim_{t \to \infty} H_i'(t) = 0$$

The above four cases of the behavior of the solutions of the respective initial-boundary value problems are also true if in Eqs. (1.5)-(1.8) one replaces the quantities H_i by $H_{i\infty}$ $(i = 1, 2)$. In the fourth case in which the inequalities $u_{*\infty} - v_\infty(x^\circ) < 0$, $u_{**\infty} - w_\infty(x^\circ) > 0$ similar to (2.11) hold, the relations (2.12) are valid and the solution tends asymptotically to the periodic (1.7) with a replacement as described above.

The effect of overflows is shown in the diagram together with the graphs of the functions

$$Y = \frac{\varphi(T_1, T_2)}{c + d} < 0, \quad Z = \frac{\psi(T_1, T_2)}{c + d} > 0$$

evaluated in accordance with (1.8) as dependent on T_1 for $x^\circ/l = 0.5$, $\Delta = T/T_1$ (continuous lines are for $b = 1/60$, dashed lines for $b = 0$). Curve 1 corresponds to $\Delta = 4$, curve 2 to $\Delta = 2$, and curve 3 to $\Delta = 4/3$.

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